15/9/23

MATH 2050A Lecture

Announcements:

- Tutorials will start next week.
- HW1 on gradescope today/Monday.

Objective:

1) Discuss some consequences of completeness ($\exists r \in \mathbb{R}$ s.t. $r^2 = 2$).
2) Discuss axioms of $\mathbb{R}$ as a whole.

Consequences of Completeness

Lem: $\forall \varepsilon > 0$, if $a \geq 0$ is s.t. $0 \leq a < \varepsilon$, then $a = 0$. $(\star)$

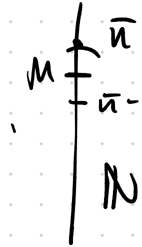
Pf: Suppose $a > 0$. Then take $\varepsilon_0 = \frac{1}{2}a > 0$. So by $(\star)$, $0 \leq a < \frac{1}{2}a$. But also,

$$0 < \underbrace{\tfrac{1}{2}a}_{\varepsilon_0} < a,$$ so we contradict $(\star)$. ✓

Thm: (Archimedean Property, A.P.) of $\mathbb{N}$: $\mathbb{N}$ is unbounded. i.e. $\forall x \in \mathbb{R}$, $\exists n_x \in \mathbb{N}$ s.t. $x < n_x$.

Pf: Suppose $\mathbb{N}$ is bounded. Then $\exists\, \bar{n} \in \mathbb{R}$ s.t. $\bar{n} = \sup \mathbb{N}$. (by completeness axiom).
(we know $1 \in \mathbb{N}$, so non-empty).

$\bar{n} - 1 \le \bar{n}$, i.e. $\bar{n}-1$ is not an upper bound of $\mathbb{N}$, i.e. $\exists\, m \in \mathbb{N}$ s.t. $\bar{n} - 1 < m$.



adding 1 on both sides gives

$$\bar{n} < m+1.$$

But by M.I. if $m \in \mathbb{N}$, then $m+1 \in \mathbb{N}$, so this contradicts the fact that $\bar{n} = \sup \mathbb{N}$.

Lem: Let $S := \{ \frac{1}{n} : n \in \mathbb{N} \}$. Then $\inf S = 0$.

Pf: Clearly $0 \le \frac{1}{n}$ for all $n \in \mathbb{N}$. So 0 is a lower bound of S.
Since $1 \in \mathbb{N}$, so $\frac{1}{1} = 1 \in S$. So $S \ne \emptyset$. So by completeness axiom, $w = \inf S$ exists.
Since 0 is a lower bound of S, we have $0 \le w$. (since w is greatest l.b.).
For any $\varepsilon > 0$. by A.P., $\exists\, n_\varepsilon \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < n_\varepsilon \Rightarrow \frac{1}{n_\varepsilon} < \varepsilon$.

Therefore, $0 \leq w \leq \frac{1}{n_\varepsilon} < \varepsilon$. Since ε was arbitrary, $w = 0$. //

<u>Cor</u>: If $x > 0$, $\exists n_x \in \mathbb{N}$ s.t. $0 < \frac{1}{n_x} < x$.

Pf: Since $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$, $x > 0$, so x is not a lower bound for S. Therefore $\exists n_x \in \mathbb{N}$ s.t. $0 < \frac{1}{n_x} < x$.

<u>Cor</u>: If $x > 0$, then $\exists n_x \in \mathbb{N}$ s.t. $n_x - 1 \leq x < n_x$.

Pf: $S = \{m \in \mathbb{N} : x < m\} \subseteq \mathbb{N}$.

1) By A.P., $S \neq \emptyset$.

2) Well-ordering property of $\mathbb{N}$, $\exists m_0 \in \mathbb{N}$ s.t. $m_0 = \min S$ $(= \inf S)$.

In particular, $m_0 - 1 \notin S$. So $m_0 - 1 \leq x < m_0$. //

Thm: (Tanh: We have filled in the gaps in $\mathbb{Q}$!") $\exists\, u \in \mathbb{R}$ s.t. $u^2 = 2$.

Pf: Let $S := \{x \in \mathbb{R} \text{ s.t. } x^2 < 2\}$.

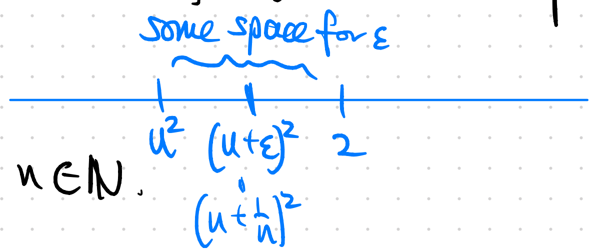
1) Clearly $1^2 = 1 < 2$, so $1 \in S$ and $S \neq \emptyset$.

2) S is bdd. from above: if $x_0 > 2$ and $x_0 \in S$, then $x_0^2 > 4 > 2$. but this contradicts $x_0 \in S$. So $x_0 \notin S$. So S is bdd. from above.

So by completeness axiom, $u = \sup S$ exists. WTS $u^2 = 2$.

We'll show this by proving $u^2 > 2$, $u^2 < 2$ are impossible.

First suppose $u^2 < 2$:



Note: $\frac{1}{n^2} \leq \frac{1}{n}$, for any $n \in \mathbb{N}$.

$$\left(u + \frac{1}{n}\right)^2 = u^2 + \frac{2u}{n} + \frac{1}{n^2} \leq u^2 + \frac{2u}{n} + \frac{1}{n} = u^2 + \frac{1}{n}(2u+1)$$

We want to choose n s.t.

$$\frac{1}{n}(2u+1) < 2 - u^2 \qquad (u^2 < 2 \Rightarrow 2 - u^2 > 0).$$

$\frac{1}{n} < \frac{2-u^2}{2u+1}$ So I want to be able to choose an $n \in \mathbb{N}$ s.t. $\leftarrow$ holds. ✓

Note: $2-u^2 > 0$, $u > 0$. (u upper bdd. for S, $1 \in S$, so $u \geq 1 > 0$).

So $\frac{2-u^2}{2u+1} > 0$. So by A.P. $\exists n_0 \in \mathbb{N}$ s.t. $0 < \frac{1}{n_0} < \frac{2-u^2}{2u+1}$.

So that means for this particular n_0

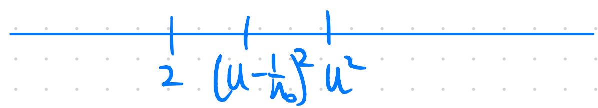
$$\left(u+\frac{1}{n_0}\right)^2 = u^2 + \frac{2u}{n_0} + \frac{1}{n_0} \leq u^2 + \frac{2u}{n_0} + \frac{1}{n_0} = u^2 + \frac{1}{n_0}(2u+1)$$

$$< \cancel{u^2} + \frac{2-\cancel{u^2}}{\cancel{2u+1}}\cancel{(2u+1)} = 2.$$

So $u + \frac{1}{n_0} \in S$. But this contradicts the fact that u is an upper bound for S.

So we've shown that $u^2 < 2$ is impossible.

Now suppose $u^2 > 2$.



By A.P. $\exists n_0 \in \mathbb{N}$ s.t. $\underbrace{\frac{1}{n_0} < \frac{u^2-2}{2u}}$ $\left(\begin{array}{l} u^2>2 \Rightarrow u^2-2>0 \\ u>0 \end{array}, \text{ so } \frac{u^2-2}{2u}>0 \right)$.

Then $\left(u-\frac{1}{n_0}\right)^2 = u^2 - \frac{2u}{n_0} + \frac{1}{n_0^2} > u^2 - \frac{2u}{n_0} > \cancel{u^2} - \cancel{2u}\left(\frac{\cancel{u^2}-2}{\cancel{2u}}\right) = 2.$

So if $s \in S$, then $s^2 < 2 < \left(u-\frac{1}{n_0}\right)^2 < u^2$, so $u-\frac{1}{n_0}$ is an upper bound of S.

But this contradicts the fact that u is the least upper bound.

So $u^2>2$ is impossible. $\Rightarrow u^2=2$ ✓.

Recall: $\sqrt{2} = 1.41421356237\ldots$

$1, \; 1.4, \; 1.41, \; 1.414, \ldots \rightsquigarrow \sqrt{2}$

these are all rational numbers. "approximates"

Expectation: $\forall x \in \mathbb{R}$, $\exists$ "sequence of $q_n \in \mathbb{Q}$ s.t. q_n approximates $\sqrt{2}$ as $n \to \infty$"

Density in $\mathbb{R}$

Thm (Density of $\mathbb{Q}$ in $\mathbb{R}$): $\forall x < y \in \mathbb{R}$, $\exists q \in \mathbb{Q}$ s.t. $x < q < y$.

Pf: Since $y - x > 0$, by A.P. $\exists m \in \mathbb{N}$ s.t. $y - x > \frac{1}{m} > 0$.

$\Rightarrow my > 1 + mx$.

Cor. above $\Rightarrow \exists n \in \mathbb{N}$ s.t. $n - 1 \leq mx < n$.

Combining above two inequalities, $mx < n \leq mx + 1 < my$

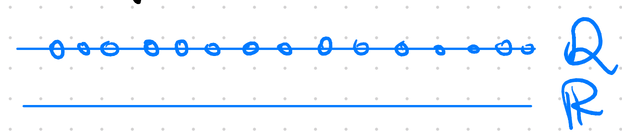
So $q := \frac{n}{m}$ satisfies $x < \frac{n}{m} < y$. //

Meaning: Say $\sqrt{2} = x \in \mathbb{R}$.

for any $n \in \mathbb{N}$, $\sqrt{2} - \frac{1}{n} = y \in \mathbb{R}$, then the thm says $\exists q_n \in \mathbb{Q}$ s.t.

$$\sqrt{2} - \frac{1}{n} < q_n < \sqrt{2}.$$

$\Rightarrow$ "$q_n \to \sqrt{2}$ as $n \to \infty$".



Thm (Density of $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$): $\forall x, y \in \mathbb{R}$ with $x < y$, $\exists r \in \mathbb{R}\setminus\mathbb{Q}$ s.t.

$$x < r < y.$$

Pf: $\exists q \in \mathbb{Q}$ s.t. $\sqrt{2}x < q < \sqrt{2}y$. (by density of $\mathbb{Q}$ in $\mathbb{R}$).

$\Rightarrow x < \frac{q}{\sqrt{2}} < y$ and we know $\frac{q}{\sqrt{2}} \in \mathbb{R}\setminus\mathbb{Q}$ since $\sqrt{2} \in \mathbb{R}\setminus\mathbb{Q}$.

Alternatively ($x - \sqrt{2} < q < y - \sqrt{2}$

$\Rightarrow x < q + \sqrt{2} < y$.)

Informally, can think of "$\mathbb{R} = \{$ limit of $\mathbb{Q} \}$."

Discussion of Axioms of $\mathbb{R}$:

1) Algebraic Axioms: (a1) $a + b = b + a$

(a2) $a + (b + c) = (a + b) + c$

(a3) $\exists\, 0 \in \mathbb{R}$ s.t. $a + 0 = a$

(a4) $\forall a \in \mathbb{R}, \exists b \in \mathbb{R}$ s.t. $a+b=0$.

(m1) $ab = ba$

(m2) $a(bc) = (ab)c$

(m3) $\exists 1 \in \mathbb{R}$ s.t. $a1 = a$.

(m4) $\forall a \in \mathbb{R}, a \neq 0, \exists b \in \mathbb{R}$ s.t. $ab = 1$.

(d) $a(b+c) = ab + ac$.

Any non-empty set F that satisfies axioms (a1) – (d) is called a <u>field</u>.

<u>Uniqueness of inverses and Identity</u>:

Thm (1) if $a, b \in \mathbb{R}$ s.t. $a+b=a$, then $b=0$.

(2) if $a, b \in \mathbb{R}$ s.t. $ab = a$, then $b=1$

(3) Given $a \in \mathbb{R}$, if $b, c \in \mathbb{R}$ s.t. $a+b = a+c$, then $b=c$.

(4) Given $a \in \mathbb{R}$, if $b, c \in \mathbb{R}$ s.t. $ab = ac$, then $b = c$.

Pf: (1): We have $a+b=a$. By (a4) $\exists a'$ s.t. $a+a' = 0$.

Then $(a+b)+a' = a+a' = 0$

By (a1), (a2), $(a+b)+a' \overset{(a2)}{=} a+(b+a') \overset{(a1)}{=} a+(a'+b)$

$\overset{(a2)}{=} (a+a')+b = 0+b \overset{(a3)}{=} b$

So $b=0$.

(2): We have $ab=a$. By (M4), $\exists\, a'$ s.t. $aa'=1$.

$(ab)a' = aa' = 1$

$LHS = (ab)a' \overset{(m2)}{=} a(ba') \overset{(m1)}{=} a(a'b) \overset{(m2)}{=} (aa')b = 1\cdot b \overset{(m3)}{=} b$

So $b=1$.

(3); (4): Left as exercise.

2) Ordering Axiom: set of positive numbers in $\mathbb{R}$.

3) Completeness Axiom: supremums exist.

Def: $\mathbb{R}$ is a complete ordered field.

Q: Are there any other complete ordered fields?

Answer: No: If K is a complete ordered field, then $\exists\, \Phi: \mathbb{R} \to K$ s.t.

Φ is a bijection that preserves field axioms and order.

$K \cong \mathbb{R}$

isomorphic.